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Energy–time uncertainty relations and time operators

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It is proved that for any Hamiltonian in a separable Hilbert space, having a non-empty absolutely continuous spectrum, there exists a time operator densely defined in the subspace of absolutely continuous vectors. This result is obtained by using the Carbó-Dorca parameterized vector spaces and the spectral representation theorem for self-adjoint operators in Hilbert spaces. The restriction of the Hamiltonian to the absolutely continuous subspace and its time operator are incompatible. These results bring a completely new light on the energy–time uncertainty relations. The possible physical interpretations and related facts are also examined.

KEY WORDS: uncertainty relations, time operators, energy–time uncertainty relations, cyclic operators, parameterized vector spaces

1. Preliminaries

This work is about the old and quite controversial problems of the energy–time uncertainty relations (ETUR) and the existence of time operators as well defined mathematical objects with an acceptable physical interpretation. Before beginning the presentation of the problem, it would be useful to know several authorized opinions about ETUR. For instance, in a private communication (1983), Hooker advanced the opinion that "the energy–time relations are quite mysterious in quantum mechanics" [1]. Aharonov and Bohm [2, 3] and later Busch [4] concluded that "the interpretations of the ETUR are in fact untenable". In the same paper Busch also affirms that "today one finds physicists claiming that there is no energy–time uncertainty relation at all".

Closely related with ETUR is the problem of the existence of a time-operator as a physical observable. It can be hypothesized, taking into account the ETUR, that the time operator should be the companion of Hamiltonians in such relationships. Starting with this hypothesis Pauli proves that it is impossible to have a time operator corresponding to "universal time". His argument uses the assumption that the spectrum of such an operator must be the set $\mathbb{R}$ of all real numbers (here the analogy with the "time-parameter" is obvious). On

the other hand, as a time operator is the companion of energy in an uncertainty-type relationship, its spectrum has to coincide with the spectrum of any Hamiltonian. In conclusion, the spectrum of any Hamiltonian must be $\mathbb{R}$, which is obviously absurd. One can use such Pauli reduction to absurd as a non-existence proof of ETUR. For discussing this and other related problems, several mathematical notions will be introduced and will be also used in what follows.

By $\mathbf{H}$ will be denoted an infinite-dimensional separable Hilbert space, whose normed vectors represent the pure states of the attached systems. For arbitrarily given $x, y \in \mathbf{H}$ $\langle x, y\rangle$, $\|x\| = \langle x, x\rangle^{1/2}$, respectively represent the inner product and the norm of the specified vectors. The symbols $\mathbb{C}, \mathbb{N}$ denote the sets of complex and natural numbers, respectively. Returning to the initial problem, one can remember that the most general uncertainty relationship involving the observables/self-adjoint operators A and B in the Robertson–Schrödinger version is:

$$\Delta_x A \Delta_x B \geq \frac{1}{2}\left|\sqrt{\langle [A, B]\rangle_x}\right|, \tag{1}$$

where $\langle C\rangle_x = \langle x, Cx\rangle$, $\Delta_x C = \sqrt{\langle C^2\rangle_x - \langle C\rangle_x^2}$ are the mean and the standard deviation of the observable C in the state x. Since, as Pauli concluded, a time operator does not exist, the usually called ETUR is simply a relationship between two quantities, which has nothing to do with the "true" uncertainty relationships involving two observables.

Hilgevoord [5] presented a completely different but quite natural point of view on this problem; affirming, in essence, that time operators exist, but they are to be defined for each dynamical system separately. In Hilgevoord's framework it is also admitted that not any dynamical system possess a time operator. In this work Hilgevoord's arguments will be not be further discussed, because they have not very much in common with the present development. But it is important to note that the here advanced arguments arrive to the same general idea, although following substantially different ways.

The present approach is based and at the same time justified by some preliminary observations on uncertainty relationships. Starting with the general relation (1) it will be only discussed the case of the observables whose spectrum has a discrete part (the set of all eigenvalues) and an absolutely continuous part (these notions will be fully explained in Paragraph 2). In fact, the most physically interesting Hamiltonians are associated to this situation. It is very important to understand that, besides some specific cases, the general relationship has not exactly the physical significance of position–momentum uncertainty relations. Indeed, first of all the position–momentum expresses a strong correlation between two observables, each of them being of fundamental importance in order to describe a large class of quantum systems. Referring to one-dimensional systems only (considering more than one dimension is irrelevant to our discussion) one can see that the position observable q and

the momentum p of a particle are, each of them, complete systems of observables in Dirac sense [6]. This means that any state of a "one-dimensional particle" is a square integrable function of q or p, *but not of both*. It is well known that this fact is, in some extent, a consequence of the uncertainty principle, which, roughly speaking, asserts that, *independently of a one-particle system state, the position and the momentum have not simultaneously determined values.* In fact, they have not determined values at all. Indeed, from the statistical nature of the standard quantum mechanical formalism, one knows that *in any arbitrarily chosen state of a particle, the probability that the particle has a well-defined position or momentum is zero.* The mathematical form of the uncertainty principle is the relation (1) for the pair (q, p). There are some additional mathematical properties of the observables q and p, which are important for the present development. One of the most significant can be stated as the ranges of the spectral measures of the observables p and q are maximal non-atomic Boolean algebras of orthogonal projectors in the L^2-Hilbert space of one-particle wave functions. In the usual quantum-mechanical language this means that the standard deviations of q and p in any state possess non-vanishing values. Or, equivalently, q and p have not eigenvalues. That shows why the relation (1) for the pair (q, p) is so important: *it reflects a strong correlation between experimentally obtained values of q and p in any arbitrarily given state.* However, this may be not the case when other pairs of observables are considered. For instance, if in equation (1) x is an eigenstate of A, then $\Delta_x A = 0$ and no information is obtained about $\Delta_x B$. Such observations suggest that the pair (q, p) plays a special role among pairs of incompatible observables and this seems to be related with the fact that they have a purely absolutely continuous spectrum. It will be proved that this is indeed so. To be more exact, it will be shown that, in essence, an observable A is involved in a uncertainty-like relation if another observable B can be described, such that the pair (A, B) is unitarily equivalent – in a sense, which will become clear below– with the "one-dimensional" pair (q, p). This fact is crucial for obtaining a meaningful interpretation of ETUR.

The present approach of ETUR was substantially suggested by the Carbó-Dorca derivation of Heisenberg uncertainty relations in the framework of parameterized n-dimensional spaces [7,8], introduced by himself. The mathematical details are very clearly exposed in the mentioned papers, so that here we present only some basic notions. A parameterized n-dimensional space is a collection $\mathfrak{R}$ of objects $X = (x_1, \ldots, x_i, \ldots, x_n)$, where $x_i, 1 \leq i \leq n$, are complex functions defined on a domain D of real numbers, which may be called components of X. It is a vector space with the algebraic operations $X+Y$ and $\lambda X, \lambda \in \mathbb{C}$, defined in the standard manner. Obviously, this is possible if for any fixed $i, 1 \leq i \leq n$, the components x_i of all X run a complex vector space of functions. In the space $\mathfrak{R}$ is defined a so-called real inward scalar product by the formula:

$$\langle X : Y \rangle = \frac{1}{2} \sum_i \int_D [x_i(t)^* y_i(t) + y_i(t)^* x_i(t)] \mathrm{d}t \in \mathbb{R}.$$

The norm generated by this scalar product is, for each component of X, that of a L^2-space of complex functions. The mathematical machinery developed by Carbó-Dorca in this structure is based on the notion of a *triad*. If the linear operators τ and ∂_t on $\mathfrak{R}$ are introduced by the formulas: $[\tau X](t) = (tx_1(t), \ldots, tx_N(t))$, $[\partial_t X](t) = (\frac{\mathrm{d}}{\mathrm{d}t}x_1(t), \ldots, \frac{\mathrm{d}}{\mathrm{d}t}x_n(t))$, then a triad is a triple $(X, \tau X, \partial_t X)$. Finally, it is proved that for any triad, which satisfies some natural properties, a Heisenberg uncertainty relationship of the form:

$$\Delta_X(\tau)\Delta_X(\partial_t) \geq \frac{1}{2} \tag{2}$$

holds, $\Delta_X(\tau)$ and $\Delta_X(\partial_t)$ being the standard deviations calculated for the vector X. In the present work, the operator ∂_t will be called the companion of τ in the parameterized vector space $\mathfrak{R}$ (in the original Carbó-Dorca papers the notion of companion has another sense).

The following material is structured in two principal parts. In the first part of the second section is described the spectral representation theorem in Hilbert spaces as a purely mathematical result, because some details of its proof are necessary for a better understanding of physical considerations. Then it is shown that this theorem permits the natural construction of a parameterized vector space for any self-adjoint operator with some convenient properties. In the third section, ETUR and the time operators are discussed from both mathematical and physical points of view. In the final section of comments there are presented some general quantum aspects, which are not directly related with ETUR, but bring some light on an appropriate understanding of their quantum formalism.

2. Spectral representation theorem

Let A be a self-adjoint operator in the separable Hilbert space **H.** According to the von Neumann spectral theorem, A is completely determined by its spectral family $\{P_\lambda; \lambda \in \mathbb{R}\}$ of orthogonal projectors, in the sense that: $\langle x, Ay\rangle = \int \lambda \mathrm{d}\langle x, P_\lambda y\rangle, \forall x, y$, [9]. It is useful to remember that the mentioned spectral family – as any spectral family – generates on the set **B** of all Borel subsets of $\mathbb{R}$ a spectral measure, that is: a projector valued σ-additive function P, having also the properties: $P(\emptyset) = 0, P(\mathbb{R} - B) = P(B)^\perp \equiv 1 - P(B), \forall B \in \mathbf{B}$, such that $P_\lambda = P((-\infty, \lambda)), \forall\lambda$. It is obvious that, given $x \in \mathbf{H}$ arbitrarily fixed, the correspondence: $B \mapsto \langle x, P(B)x\rangle \equiv P_x(B)$, defines a real measure on **B**. If x is normed, then P_x is a probabilistic measure.

Here begins the presentation of the spectral representation theorem. It must be noted that this exposition uses consistently certain basic developments from the Gelfand and Vilenkin's book about generalized functions [10]. The central notion is the one of a cyclic operator. The operator A is said to be cyclic if

there exists $x \in \mathbf{H}$ such that the subspace generated by the set $\{P(B)x;\, B \in \mathbf{B}\}$ is dense in $\mathbf{H}$. The vector x is also called cyclic vector. The following assertion – named as the spectral representation theorem here – is the basic mathematical fact in the present development. It can be stated as: *if A is a cyclic operator with the cyclic vector x, then* $\mathbf{H}$ *is isomorphic to the space* $L^2(\mathbb{R}, \mathbb{C}, P_x)$ *defined on* $\mathbb{R}$ *of complex square integrable functions with respect to* P_x *. If* $U : \mathbf{H} \to L^2(\mathbb{R}, \mathbb{C}, P_x)$ *is the corresponding isomorphism and introducing the notations:* $A_U = UAU^{-1}, x_U = Ux$, *then* $(A_U x_U)(\lambda) = \lambda x_U(\lambda)$. In other words, the isomorphism U transforms the operator A into a "coordinate" operator over a space of square integrable functions of one real variable. Assuming that A has a purely absolutely continuous spectrum means that the measure P_x is absolutely continuous with respect to the Lebesgue measure ν, that is: $\nu(B) = 0 \Rightarrow P_x(B) = 0$. Obviously, $L^2(\mathbb{R}, \mathbb{C}, P_x)$ is a one-dimensional parameterized vector space with the scalar product defined by the measure P_x. This fact makes it different from the Carbó-Dorca's parameterized spaces, which are defined by the Lebesgue measure. Therefore, the companion of A_U in $L^2(\mathbb{R}, \mathbb{C}, P_x)$ has to be found – it can be denoted by ∂ – such that an uncertainty relation for the pair (A_U, ∂) might be derived by using the procedure from the paper [7]. If that proof is carefully examined, then it will become clear that the step in which the measure is consistently involved is the verification of the sequence of equalities:

$$\int_{-\infty}^{\infty} \frac{\mathrm{d}}{\mathrm{d}t}(t\,|x(t)|^2)\mathrm{d}t = [t\,|x(t)|^2]_0^{\infty} = 0. \tag{3}$$

Clearly, it is valid since the function $|x(t)|^2$ vanishes rapidly enough when $t \to \infty$ since $x(t)$ is in the domain of τ.

Looking for the operator ∂ and taking into account the just mentioned proof, it becomes clear that it must be a sort of derivative, because the properties of the derivatives are consistently used. There will be explained first, as clear as possible, how the operator ∂ may be constructed in a quite natural way. In order to simplify the notation it can be written: $P_x \equiv \mu$. The measure μ being absolutely continuous with respect to Lebesgue measure, one may formally write $\mathrm{d}\mu(\lambda) = \omega(\lambda)\mathrm{d}\lambda$ – where ω is a positive Lebesgue integrable function – in the right part it is used the "common" notation for the Lebesgue measure. Such a relation expresses in fact the well-known Radon–Nicodym theorem. In other words, it may be written for any Borel subset $B \subseteq \mathbb{R}$ the equality: $\mu(B) = \int_B \omega(\lambda)\mathrm{d}\lambda$. Taking into account this fact, one can define the function $m : \mathbb{R} \to \mathbb{R}, m(\lambda) = \mu((-\infty, \lambda))$, which is increasing and almost everywhere derivable. In what follows one can consider that m is strictly increasing and derivable (anyway, it is almost everywhere derivable). Of course, one may envisage more general cases, but the purpose here is to discuss the physical part of the problem and not the possible mathematical difficulties, which might appear in some specific

cases. So, in the adopted framework it can be written: $m'(\lambda) = \omega(\lambda)$, where f' denotes the derivative of f. Similarly, one has: $\mathrm{d}m(\lambda) = \omega(\lambda)\mathrm{d}\lambda$.

Now, there are present all necessary tools for defining the operator ∂, i.e. such an operator, which replaces the derivative in order to obtain the analogue of the relation (3)in the space $L^2(\mathbb{R}, \mathbb{C}, \mu)$. In order to do this, one can define the derivative of a function with respect to a strictly increasing absolutely continuous function. Such a function is m, so that one can say that the function f is derivable in λ with respect to m if

$$\lim_{h\to 0} \frac{f(\lambda + h) - f(\lambda)}{m(\lambda + h) - m(\lambda)}$$

exists. Then one can denote this limit by $f'_m(\lambda)$ or $\frac{\mathrm{d}f}{\mathrm{d}m}(\lambda)$. It is not useless now to observe that the usual derivative of f is obtained when, instead of m, one takes the identity of $\mathbb{R}$.

Everything is prepared now for defining the operator ∂; it must be: $\frac{\mathrm{d}}{\mathrm{d}m} = \frac{1}{\omega}\frac{\mathrm{d}}{\mathrm{d}\lambda}$. Within these conditions the operator is well defined, since m is strictly increasing, which means that ω is strictly positive. All steps for deriving the general Heisenberg relationship (from the paper [7]) may be easily reproduced in $L^2(\mathbb{R}, \mathbb{C}, P_x)$ by using this operator. Now it is clear that in $L^2(\mathbb{R}, \mathbb{C}, \mu)$ the equality (3) becomes

$$\int_0^\infty \frac{\mathrm{d}}{\mathrm{d}m}(\lambda\,|y(\lambda)|^2)\mathrm{d}\mu = \int_0^\infty \frac{1}{\omega(\lambda)}\frac{\mathrm{d}}{\mathrm{d}\lambda}(\lambda\,|y(\lambda)|^2)\omega(\lambda)\mathrm{d}\lambda = \Big[\lambda\,|y(\lambda)|^2\Big]_0^\infty.$$

For instance, the condition $[\lambda\,|y(\lambda)|^2]_0^\infty = 0$ is satisfied for all elements of $C_0^\infty(\mathbb{R})$, i.e. for all smooth finitely supported functions. Moreover, since in what follows only operators whose absolutely continuous spectrum is the interval $[0, \infty)$ will be considered, we have to admit that, even if the measure μ is finite, the functions in the domain of A_U vanish rapidly enough when $\lambda \to \infty$. It results finally that the pair (A_U, ∂) satisfies an uncertainty relation formally similar with (2).

In order to obtain a general result, that is: the fact that (2) holds for a self-adjoint operator A, whose spectrum is absolutely continuous but it is not cyclic, one has to follow a mathematical route, essentially based on the just examined case of cyclic operators. As it has been already said, the idea of proof emerges directly from the Gelfand development in reference [10]. However, the mentioned mathematical development is modified and completed for including all details, which are essential for completing and clarifying the physical facts one is interested in. So, consider again A, a self-adjoint operator in $\mathbf{H}$ with the domain $D(A)$. A is supposed to have a purely absolutely continuous spectrum. Besides, A is not assumed to be cyclic. If $x \in D(A) \subseteq \mathbf{H}$ is an arbitrarily given vector, then

the smallest closed subspace containing all vectors of the type $P(B)x$ – where B runs the set of all Borel subsets of $\mathbb{R}$ – is called the cyclic subspace generated by x, and will be denoted by H_x. The subspace H_x is invariant under A and the restriction A^x of A to H_x is a cyclic operator. In order to prove that H_x is invariant under A, it will be verified first that: $Ax \in H_x$. Suppose that $\langle y, P(B)x\rangle = 0$ for all $P(B)$. From the spectral theorem one knows that: $\langle y, Ax\rangle = \int \mathrm{ad}\,\langle y, P_a x\rangle$, where the integral is taken with respect to the complex measure $B \mapsto \langle y, P(B)x\rangle$. But this measure is zero for all B, so that: $\langle y, Ax\rangle = 0$. Therefore, Ax is orthogonal to any vector belonging to the orthogonal subspace of H_x, which means that:$Ax \in H_x$. It follows that: $AP(B)x = P(B)Ax \in H_x$ for all B, which proves the required invariance. Further, the subspace H_x cannot be finite dimensional, because in this case A^{x} would have eigenvectors. Since any eigenvector of A^{x} is also an eigenvector of A, one arrives to the absurd conclusion that A has not a purely continuous spectrum. Further, it is trivially verifiable that A^{x} has a purely absolutely continuous spectrum. Indeed, it is sufficiently to see that all vectors of H_x are absolutely continuous, [11]. Therefore, it becomes clear that, if $\mu_x(B) = \langle x, P(B)x\rangle$, then H_x is isomorphic to $L^2(\mathbb{R}, \mathbb{C}, \mu_x)$. That is, because x is a cyclic vector for A^{x}. Here and below it is important to remember that the elements of the space $L^2(\mathbb{R}, \mathbb{C}, \eta)$ are not functions, but classes of η-equivalent functions (f and g are said to be η-equivalent if $\eta(\{x \in \mathbb{R};\, f(x) \neq g(x)\}) = 0$). Therefore, it may be considered that a square integrable with respect to η function vanishes outside the support of η.

Now it will be proved that there exists a finite or denumerable family of mutually orthogonal cyclic subspaces, whose direct sum is the whole space $\mathbf{H}$. A simple observation, which will be consistently used, is that the cyclic subspaces generated by two orthogonal vectors are orthogonal. The proof is straightforward. Consider now an orthonormalized basis $B_A = \{x_n; n \in \mathbb{N}\}$, contained in the domain of A and denote by H_1 the cyclic subspace generated by x_1. If $\mathbf{H} \neq H_1$, then suppose $x_{k_2} \in B_A$, is the first vector, which do not belongs to H_1. Let H_2 be the cyclic subspace generated by the vector:x_{k_2}. Obviously, the cyclic subspaces H_1, H_2 are orthogonal. Further, if $\mathbf{H} \neq H_1 \oplus H_2$ then one can take: $x_{k_3} \in B_A$, as the first vector, which is not in $H_1 \oplus H_2$, naming H_3 the cyclic subspace generated by this vector. It is clear that this process stops after a finite or countable number of steps. Since the two cases are not essentially different, it will be further considered that

$$\mathbf{H} = H_1 \oplus H_2 \oplus \cdots \oplus H_n = \bigoplus_{i=1}^{n} H_i. \tag{4}$$

Taking into account the known fact that each H_i is isomorphic to the function space: $L^2(\mathbb{R}, \mathbb{C}, \mu_i) \equiv L_i^2$, where μ_i is the measure defined by the corresponding cyclic vector to H_i. Let $U_i : H_i \to L_i^2$ be the canonical isomorphism between the specified subspaces. Given $x \in \mathbf{H}$, from (4) it results that there exists a unique representation:

$x = x^1 + x^2 + \cdots + x^{\mathrm{n}}$, with $x^i \in H_i$, $1 \leq i \leq n$. Then, the isomorphism U: $\mathbf{H} \rightarrow \overset{n}{\underset{i=1}{\oplus}} L_i^2$ may be defined by the equality: $U(x) = (U_1(x^1), \ldots, U_n(x^{\mathrm{n}})) \equiv (x^1_{U_1}, \ldots, x^{\mathrm{n}}_{U_n})$. One shall observe that the space $\overset{n}{\underset{i=1}{\oplus}} L_i^2$ is a n-dimensional parameterized space with its elements being of the form: $(f^1, \ldots, f^{\mathrm{n}})$, where $f^i \in L_i^2$, $1 \leq i \leq n$. So, one can now define the operator A_U and its derivative/Heisenberg companion ∂ by the formulas:

$$[A_U(x^1_{U_1}, \ldots, x^{\mathrm{n}}_{U_n})](\lambda) = (\lambda x^1_{U_1}(\lambda), \ldots, \lambda x^{\mathrm{n}}_{U_n}(\lambda)),$$

$$[\partial(x^1_{U_1}, \ldots, x^{\mathrm{n}}_{U_n})](\lambda) = (\frac{\mathrm{d}}{\mathrm{d}m_1} x^1_{U_1}(\lambda), \ldots, \frac{\mathrm{d}}{\mathrm{d}m_n} x^{\mathrm{n}}_{U_n}(\lambda)).$$

The pair (B_A, U) will be called a Heisenberg representation for the operator A and (A_U, ∂), the Heisenberg pair associated to this representation. For such a pair there exists a generalized Heisenberg relationship, as it has been proved by Carbó-Dorca [7, 8].

At the end of this paragraph, a technical but very important observation has to be put forward. It refers to the fact that for a given self-adjoint operator, there are in general infinitely many Heisenberg representations. To see this, consider the example of a cyclic operator A and an orthonormalized basis containing a cyclic vector x. Starting the construction of a Heisenberg representation with x, then the obtained representation will be in a one-dimensional parameterized space. Taking first a vector from such a basis, which is not cyclic, then the resulting representation will be in a parameterized space of more than one dimension. Presently, the mathematical details of this problem, which are surely not so simple, will not be discussed. It is important to note there it is natural to assume that *for any physically significant operator/observable, it is possible to construct a physically consistent Heisenberg representation. If the operator is cyclic, the parameterized space of the representation is one-dimensional, in the other situations it has more than one dimension.* The only general criterion for choosing a Heisenberg representation is its simplicity. That is why for the case of a cyclic observable one can assume that the representation, which must be chosen, corresponds to a one-dimensional parameterized space.

3. Time operators

The central physical aim of this work is to discuss the existence of energy-time uncertainty relations where a time observable is directly involved. All necessary mathematical material for this purpose has been already developed in the preceding section. The general observable A will be replaced with a Hamiltonian $\widehat{H}$. The Hamiltonian $\widehat{H}$ is assumed to have a point spectrum σ_{p}, which may be empty, and a purely absolutely continuous spectrum σ_{ac}. It is well known that

the Hilbert space **H** may be represented as the direct sum $\mathbf{H} = \mathbf{H}_{\mathrm{p}} \oplus \mathbf{H}_{\mathrm{ac}}$, where the two terms of the sum are respectively the subspace generated by the eigenvectors and the subspace of absolutely continuous vectors of $\widehat{H}$. These subspaces are orthogonal and invariant under $\widehat{H}$ [11]. The most important part of the present discussion refers to the situation in which the restriction of $\widehat{H}$ to the absolutely continuous subspace is cyclic. That is so because all important physical problems appear in this simple context.

Consider then $\widehat{H}$ a cyclic Hamiltonian in **H,** having a purely absolutely continuous spectrum. Denote by L^2_μ the space isomorphic to **H**, the space of square integrable functions with respect to the measure μ. The measure μ is generated, as it is already known, by a cyclic vector and the spectral measure of $\widehat{H}$. It is also known that μ is absolutely continuous with respect to Lebesgue measure, so that we may write $\mathrm{d}\mu(\varepsilon) = \omega(\varepsilon)\mathrm{d}\varepsilon$, where $\omega(\varepsilon)$ is a positive Lebesgue-integrable function. If $U : \mathbf{H} \to L^2_\mu$ is the canonical isomorphism, then U and a cyclic vector define a Heisenberg representation of $\widehat{H}$, in which the companion of $\widehat{H}_U$ is the operator: $\partial = \frac{1}{\omega(\varepsilon)} \frac{\mathrm{d}}{\mathrm{d}\varepsilon}$. The operator: $T_U = \frac{h}{i}\partial$ is called the time operator associated to $\widehat{H}_U$ in this representation (where h is the Planck's constant and i the imaginary unit). The pair $(\widehat{H}_U, T_U)$ satisfies the Heisenberg uncertainty relation in the space L^2_μ (obviously, all quantities entering this relation are calculated with respect to the measure μ). It is not a problem to see that the average values of T_U have time dimensions. Since it is defined by a Hamiltonian, it will be called Hamiltonian time.

Concerning the essential properties of T_U, it is obvious that they are those of the derivative operator. Besides, if ν denotes the Lebesgue measure, one can easily verify that: $L^2_\nu \subseteq L^2_\mu$. This results from the following sequence of implications:

$$f \in L^2_\nu \Rightarrow \int |f|^2 \mathrm{d}\mu = \int |f|^2 \omega \mathrm{d}\nu \leq M \int |f|^2 \mathrm{d}\mu < \infty \Rightarrow f \in L^2_\mu,$$

where it has been used the boundedness of the function ω. Therefore, the domain of the derivative operator is included in L^2_μ. Moreover, it is clear enough, without completing the mathematical details, that the time operator can be constructed from a purely absolutely continuous operator.

It remains to see what might be the physical interpretation of the so-defined time operator. For better understanding this quite delicate matter, one can first consider a one-dimensional particle with p and q as its momentum and coordinate operators, respectively. It is known that the states of such system may be expressed as normed functions of their position or equally, as normed functions of their momentum. One can say that p and q are "state variables". This results from the fact that both p and q are cyclic operators, so that the state functions are defined on the spectrum of one of them. In this situation one must remember that the sets $\{p\}$ and $\{q\}$ are, each of them, complete sets of observables in the Dirac sense. Returning to the pair $(\widehat{H}_U, T_U)$ one can see that it reproduces,

in some sense, the case of the pair $(q,\ p)$ for one-dimensional systems. Indeed, the energy operator is cyclic and appears instead of the coordinate operator. Analogously, the time operator plays the role of the momentum operator. This parallel situation leads directly to the fact that the Hamiltonian time is a state variable, just as the energy associated to its generating Hamiltonian. More precisely, the states of the system described by the Hamiltonian in question may appear as functions of the values of the Hamiltonian time. Therefore, the Hamiltonian time and the usual "parameter time" are *objects with completely different meanings*. Indeed, the parameter time describes the change of states, i.e. what is commonly called the dynamics of the system. In other words: it is not in any case a state variable. On the other hand, the Hamiltonian time is closely related with the states of the system themselves. This means that, in an appropriate representation, it becomes a state variable. Nevertheless, being both of them "times" it is natural to ask if they have something in common. Since the time operator is an observable, one can talk about its average value in any state (from its domain). It seems that this value has not a clear enough meaning in terms of the usual "clock-time". But, following the common interpretation of average values, it represents the most probable value of the Hamiltonian time in any given state. Therefore, the standard deviation of the time operator in a state seems to be a good candidate for the lifetime of a system in that state. In other words, one can consider that the standard deviations of the time operator in different states correspond to the length of some clock-time intervals. This is clearly in accord with the Heisenberg-type relation between the energy and the Hamiltonian time in the considered Heisenberg representation. This fact gives also a clear sense to the ETUR. In the space $\mathbf{H}$ the Heisenberg pair is $(\widehat{H}, \widehat{T})$, where $\widehat{T} = U^{-1} T_U U$.

The physical interpretation of the general case, the one associated to a non-cyclic Hamiltonian, may be easily obtained. Indeed, in this situation the time operator is obviously the direct sum of the time operators defined for each cyclic component from the already known representation as a direct sum of $\mathbf{H}$.

4. Final remarks

The conclusion of the present work can be resumed by stating that: for any Hamiltonian, having an absolutely continuous spectrum, it may be constructed a time operator, acting in the absolutely continuous subspace. This operator has as main property the fact that it is the companion of its generating Hamiltonian for an ETUR, having exactly the formal aspect of the momentum-coordinate uncertainty relationship. In other words, the integral relation represented by the uncertainty relation is preserved by the pair $(\widehat{H}_U, T_U)$ in the space L^2_μ. On the other hand, this pair does not preserve the momentum-coordinate commutation relation:

$$pq - qp = \frac{h}{i} I,$$

where I is the identity operator of the corresponding Hilbert space. Indeed, if the elementary calculations are performed, then one obtains for any smooth finitely supported $f \in L^2_\mu$

$$[T_U \widehat{H}_U - \widehat{H}_U T_U] f = \frac{h}{i} \frac{1}{\omega} f.$$

Remember that $\mathrm{d}\mu = \omega \mathrm{d}\nu$ so that, when μ coincides with the Lebesgue measure, then $\omega(\varepsilon) \equiv 1$ and the Heisenberg commutation relationships are obtained. Because of that, the transfer of the commutation relations on the space **H** is difficult. Indeed, the operator corresponding to multiplication with the function $\frac{1}{\omega}$ it is not known.

The last remark one can propose is about the discrete/point spectrum of $\widehat{H}$. In the case of a Hamiltonian with purely discrete spectrum, one can not talk about ETUR in the spirit of the "classical" Heisenberg relations. Plausibly, in this case one has to choose one of the presentations existing in the literature of the "false" ETUR. In fact, most of these relationships are derived for wave functions, which may be represented as finite or infinite linear combinations of eigenfunctions of the Hamiltonians.

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